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# Minimum-Fuel Escape from Two-Body Sun-Earth System

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**Escaping from the solar system by receiving gravity assist from the Earth is considered in this paper. A simple procedure, which neglects the eccentricity of the Earth's orbit and uses the two-body problem equations and the patched-conic approximation, provides near-optimal trajectories using either a single or multiple Earth flybys. The analysis shows that the amount of propellant required to escape from the solar system decreases with the number of flybys, but the mission time increases. The same approach is also used to find near-optimal trajectories that use a single powered flyby. The eccentricity of the Earth's orbit can be exploited to reduce the characteristic velocity; an indirect optimization procedure provides the most favorable locations where the Earth should be intercepted.**

## Introduction

THE analysis of interplanetary trajectories is a complex task because of the possibility of receiving gravity assists from the planets; the increase in energy, which is obtained from a flyby, can be improved, in turn, by deep-space impulses. Therefore, multiple solutions exist to achieve, for instance, the escape from the solar system. If the two-body sun-Earth system is considered, the number of potential trajectories is reduced, and an easier insight into the problem features is possible. Under this assumption the escape trajectory can be improved only by an Earth gravity assist (EGA), which is necessarily preceded by a deep-space velocity impulse ($\Delta V$); otherwise, the magnitude of the geocentric hyperbolic excess velocity $v_\infty$ at the Earth reencounter would be the same as at departure. Under these circumstances the flyby is useless, as the optimal $v_\infty$ direction (that is, parallel to Earth's velocity) can be provided with no additional cost at departure.

The concept of $\Delta V$-EGA trajectories has been introduced by Hollenbeck[1]; a deep-space velocity impulse allows a spacecraft that has left the Earth to re-encounter it with a larger hyperbolic excess velocity ($v_\infty$ leveraging) and obtain a free gravity assist. Sweetser[2] has used Jacobi's integral for the restricted three-body problem to discuss some interesting aspects of this class of trajectories. However, the term $v_\infty$ leveraging and a systematic analysis of $\Delta V$-EGA trajectories can be ascribed to Sims et al.[3,5] and Sims and Longuski.[4] They assume that the Earth's orbit around the sun is circular and provide a clever means of finding suboptimal trajectories by nullifying thrust-misalignment losses.

Two computational procedures are used in the present analysis; both employ the two-body problem equations and the patched-conic approximation. The first procedure, which closely follows the technique proposed by Sims et al.,[5] provides near-optimal trajectories using either a single or multiple EGA. The procedure is very simple and efficient; it has been used to obtain a large part of the results presented in this paper. The eccentricity of the Earth's orbit noticeably affects the mission $\Delta V$ requirements; this is taken into account by an indirect optimization procedure that has been proposed by the authors for the analysis of interplanetary missions[6] and already applied to $\Delta V$-EGA trajectories.[7] Optimization procedures often attain solutions that are only local optima; indirect procedures usually show convergence difficulties. The authors' optimization method is not immune to these shortcomings, but these are drastically reduced if the optimization code is applied to suboptimal solutions provided by the former procedure.

The paper analyzes several strategies that allow the spacecraft to escape from the sun-Earth system; numerical results are presented. The same strategies are also a useful means of attaining a finite distance from the sun; this problem is qualitatively addressed by the accompanying figures. The presentation of many numerical results does not modify the nature of this paper, which is essentially academic; practical interplanetary missions usually receive gravity assist from the other planets in the solar system (Jupiter and Venus, mainly), which are ignored in the present analysis.

## Statement of the Problem

A spacecraft leaves the Earth and escapes from the solar system; the presence of all of the other planets is neglected, and the whole trajectory is in the ecliptic plane. The patched-conic approximation is adopted, but the radius of the Earth's sphere of influence and the time spent inside the sphere are considered to be negligible. Throughout the paper the term "escape" always refers to the solar system, whereas the term "hyperbolic excess velocity" is related only to the Earth.

Suboptimal solutions are found preliminarily by means of a very simple procedure that Sims et al.[5] have proposed to deal with simple $\Delta V$-EGA trajectories. The spacecraft (Fig. 1) is launched from the Earth with a hyperbolic excess velocity parallel to the velocity of the planet, which is assumed to move on a circular orbit around the sun. The vehicle is inserted (point 0) into a heliocentric orbit with a period that is slightly greater than an integer number of years. At the aphelion (1) thrust is again used to provide a tangential velocity impulse $\Delta v_1$ that lowers the perihelion. The spacecraft intercepts the Earth before or after the perihelion (2) with a larger and nontangential hyperbolic excess velocity that is usefully rotated by gravitation. The corresponding trajectory is not optimal, but a simple iterative algorithm is sufficient to determine the impulse $\Delta v_1$ that is necessary for the intercept. Note that the procedure is improved if the initial hyperbolic excess velocity is searched for while $\Delta v_1$ is assigned. Multiple revolutions of the Earth and spacecraft on their orbits are also possible: simple $\Delta V$-EGA trajectories are classified by means of the designation[5] $K{:}L(M)^\pm$ where $K$ is the number of Earth orbit revolutions; $L$ is the number of spacecraft orbit revolutions; $M$ is the spacecraft orbit revolution on which $\Delta v_1$ is applied; and $\pm$ indicates an Earth encounter after/before the spacecraft orbit perihelion. The number in parentheses may be omitted when the spacecraft executes only one revolution ($L = 1$).

The most interesting solutions are also analyzed by means of an optimization procedure that has been proposed by the authors[6,7] to search for minimum-$\Delta V$ (i.e., minimum-fuel) interplanetary trajectories using high-thrust constant-specific-impulse engines. This

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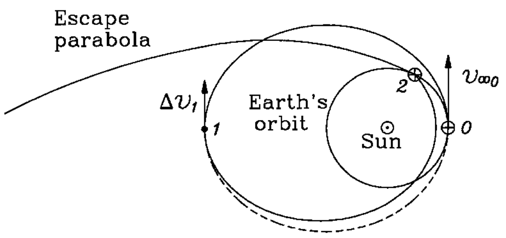


**Fig. 1 Simple Δ V-EGA trajectory.**

procedure is also based on the patched-conic approximation but removes any simplifying assumption concerning the position where the deep-space impulse is applied, its direction, and the direction of the hyperbolic excess velocity on leaving the Earth. The optimization program can also take the eccentricity of the Earth's orbit into account to quantify the performance improvement that is obtained if the spacecraft leaves and intercepts the Earth near the perihelion where the planet's velocity is higher.

The authors have already presented the necessary conditions for the optimality of a $\Delta V$-EGA trajectory whose final point is at a finite distance from the sun.[7] The flight time on the escape parabola is not finite; without any loss of accuracy, numerical difficulties are avoided by considering the last flyby as the final point of the optimization process where different conditions for optimality are required. These conditions are provided in the Appendix.

The numerical results depend on some assumptions that have been made according to Sims et al.[5] The spacecraft departs from an Earth parking orbit, which is assumed to be circular (185-km altitude) and in the ecliptic plane; the minimum allowable altitude during a flyby is 200 km.

## Escape Maneuvers

Different ways of escaping from the sun-Earth system are discussed in this section. A summary of $\Delta V$-requirements is given in Table 1. Figure 2 presents the maximum distance from the sun $r_{\max}$, which is attainable by performing similar maneuvers but applying a lower total characteristic velocity $\Delta V_{\text{tot}}$ that is not sufficient for the escape. All of these results are obtained by considering the Earth as moving at a constant velocity $v_E$ on a circular orbit and by using the simpler solution that searches for suboptimal trajectories.

### Direct Escape (DE)

In the case of a circular Earth orbit, the spacecraft directly escapes from the solar system if the departure impulse provides a hyperbolic excess velocity $v_{\infty 0} = [(\sqrt{2}) - 1]v_E = v_\infty^*$ parallel to Earth's velocity ($v_\infty^*$ is the minimum hyperbolic excess velocity that also allows the escape after an Earth flyby). This maneuver is the simplest, fastest, but most expensive way of obtaining an escape [$\Delta V_{\text{tot}} = 8.729$ km/s, as the departure impulse is applied on leaving a 185-km altitude circular low-Earth orbit (LEO)]. The flight time on the escape parabola is not finite and is conventionally assumed to be equal for all trajectories. The elapsed time $\Delta t$, from the Earth departure until the last flyby, is used throughout this paper as an index of the longer duration of more complex maneuvers.

### Simple $\Delta V$-EGA

A maneuver that uses a single deep-space velocity impulse and a single Earth flyby can achieve the escape from the solar system even though at departure $v_{\infty 0} < v_\infty^*$. The deep-space impulse ($v_\infty$ leveraging) efficiently produces a higher relative-to-Earth velocity at the reencounter ($v_{\infty 2} > v_{\infty 0}$) but also a larger angle between the directions of this velocity and the Earth's velocity; when the leveraging is increased, the rotation that is necessary to leave the Earth's sphere of influence parallel to the planet's velocity is increased. If $v_{\infty 0}$ is quite high (that is, the period of the first revolution around the sun is at least 21 years), the required leveraging and the corresponding rotation are small; the gravity assist can produce the optimal turn angle that makes $v_{\infty 2} = v_\infty^*$ parallel to $v_E$, as the hyperbola perigee is higher than the limit value. When the minimum-height constraint is reached and the turn angle is no longer optimal, the escape requires $v_{\infty 2} > v_\infty^*$. If the period of the first revolution orbit is lower than five years, the required $v_{\infty 2}$ becomes so high that it cannot be

**Table 1 Escape trajectories**

| Mission class | Initial maneuver | Number of flybys | Number of burns | $\Delta V_{\text{tot}}$, km/s | $\Delta t$,[a] years |
|---|---|---|---|---|---|
| DE | —— | 0 | 1 | 8.730 | 0 |
| $\Delta V$-EGA | $5{:}1(1)^+$ | 1 | 2 | 6.713 | 5.14 |
| $\Delta V$-PEGA | $2{:}1(1)^+$ | 1 | 3 | 7.597 | 2.14 |
| | $3{:}1(1)^+$ | 1 | 3 | 7.115 | 3.14 |
| | $4{:}1(1)^+$ | 1 | 3 | 6.868 | 4.14 |
| $\Delta V$-MEGA | $3{:}1(1)^+$ | 2 | 2 | 5.891 | 24.14 |
| | $2{:}1(1)^-$ | 3 | 2 | 5.618 | 25.81 |
| M$\Delta V$-EGA | $2{:}1(1)^-$ | 3[b] | 4 | 5.237 | 9.98 |
| | $3{:}2(1)^+$ | 3[c] | 4 | 5.036 | 11.21 |
| | $3{:}2(2)^-$ | 4[d] | 5 | 4.879 | 12.88 |
| | $5{:}4(1)^+$ | 5[e] | 6 | 4.786 | 18.18 |
| | $5{:}4(4)^-$ | 9[f] | 10 | 4.678 | 36.04 |

[a]Elapsed time from Earth departure until last flyby.
[b]$2{:}1(1)^-$ $3{:}1(1)^-$ $5{:}1(1)^+$.
[c]$3{:}2(1)^+$ $3{:}1(1)^-$ $5{:}1(1)^+$.
[d]$3{:}2(2)^-$ $2{:}1(1)^-$ $3{:}1(1)^-$ $5{:}1(1)^+$.
[e]$5{:}4(1)^+$ $3{:}2(1)^+$ $2{:}1(1)^-$ $3{:}1(1)^-$ $5{:}1(1)^+$.
[f]$5{:}4(4)^-$ $4{:}3(1)^+$ $3{:}2(2)^-$ $5{:}3(1)^+$ $2{:}1(1)^-$ $5{:}2(2)^-$ $3{:}1(1)^+$ $4{:}1(1)^-$ $5{:}1(1)^+$.

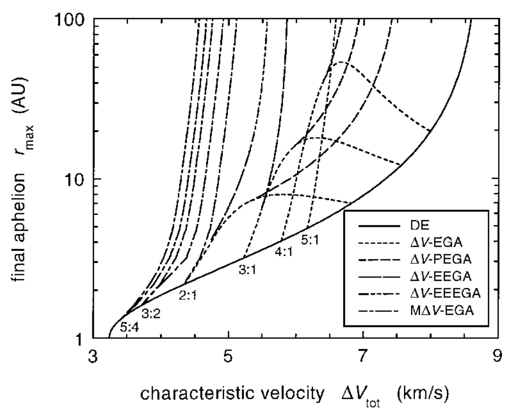


**Fig. 2 Performance of different suboptimal maneuvers based on Δ V-EGA concept (see Table 1).**

provided by the deep-space impulse, and the escape is not possible. The $5{:}1^+$ trajectory is the cheapest $\Delta V$-EGA maneuver that allows escape ($\Delta V_{\text{tot}} = 6.713$ km/s); the $5{:}1^-$ maneuver is instead preferred if the required aphelion is lower than 200 AU. The $5{:}1^+$ trajectory has been optimized by taking the eccentricity of the Earth's orbit into account; the characteristic velocity required for the escape is reduced to $\Delta V_{\text{tot}} = 6.596$ km/s.

### Powered Flyby ($\Delta V$-PEGA)

If the period of the first revolution around the sun is lower than five years, the escape could "be achieved with an additional maneuver immediately after the Earth gravity assist."[5] A velocity impulse inside the Earth's sphere of influence is instead more efficient. In a recent paper[8] the authors have shown that the thrust should be applied after the perigee and slightly misaligned with respect to the relative-to-Earth velocity to increase the turn angle. The magnitude of the impulse, its direction, and the geocentric radius where it is applied, together with the magnitude of the deep-space impulse, are obtained by means of a parameter optimization. (The simplifying assumptions concerning departure and deep-space impulses are retained to be consistent with the other results presented in Table 1 and Fig. 2.) The $\Delta V$-PEGA trajectories require a higher characteristic velocity than the $5{:}1^+$ $\Delta V$-EGA mission, but offer the advantage of shorter time-lengths (see Table 1). The authors' indirect optimization procedure cannot presently deal with a powered flyby.

### Aerogravity Assist ($\Delta V$-AEGA)

The presence of a constraint on the perigee of the planetocentric hyperbola limits the velocity turn that is provided by the flyby, thus increasing the hyperbolic excess velocity, which is required for the escape ($v_{\infty 2} > v_\infty^*$). A possible improvement is that of performing an aeroassisted Earth flyby[3]: the spacecraft is kept inside the atmosphere by aerodynamic forces until the required rotation is achieved and the escape is obtained with $v_{\infty 2} = v_\infty^*$. If an infinite lift-to-drag ratio is assumed,[3] the mission $\Delta V_{tot}$ is easily obtained by computing a simple $\Delta V$-EGA trajectory while removing the constraint on the hyperbola turn angle.

### Multiple Earth Flybys ($\Delta V$-MEGA)

The same velocity turn and, therefore, the same performance of an ideal (i.e., zero drag) $\Delta V$-AEGA maneuver can be achieved by means of a single deep-space impulse followed by a series of Earth flybys. The same nomenclature as for the corresponding $\Delta V$-AEGA maneuvers is adopted, as the trajectory is the same from departure to the first flyby and from the last flyby onward. When the height constraint prevents a single flyby from turning the hyperbolic excess velocity parallel to the Earth's velocity, the flyby is used to insert the spacecraft into an elliptical orbit, which, after an integer number of years, intercepts the Earth in the same position and with the same hyperbolic excess velocity as it had just after the previous flyby. (It is obviously possible to perform more than one revolution on an ellipse with a noninteger period, but in this case the mission time would be unusefully increased.) More than one flyby might be necessary to obtain the desired rotation; the last flyby produces the optimal $v_\infty$ direction, which is parallel to the Earth's velocity.

To save time, the turn angles of the last flybys should be close to the limit value, whereas the residual rotation should be obtained from the first flyby. A $2{:}1^-$ $\Delta V$-EEEGA trajectory with three Earth flybys and a time-length of about 26 years is the cheapest solution ($\Delta V_{tot} = 5.618$ km/s); the $3{:}1^+$ $\Delta V$-EEGA maneuver is shorter (24 years) and requires only two gravity assists with a slightly greater characteristic velocity ($\Delta V_{tot} = 5.891$ km/s).

Mission $2{:}1^-$ is presented in Fig. 3; the Earth's angular position is the same for the three successive flybys. Figure 4 presents the same trajectory, as seen in a reference frame centered on the sun and rotating with the Earth, in order to point out the direction of the relative-to-Earth velocity. One can see that the hyperbola turn angle during the first flyby is the lowest. The spacecraft finally leaves the Earth parallel to the planet's velocity.

The performances of these suboptimal maneuvers are slightly improved by the optimization procedure. However, an analysis of the solutions on the basis of Pontryagin's Maximum Principle suggests adding a deep-space impulse in the proximity of an aphelion transit between two consecutive flybys: each flyby offers the possibility of exploiting the $v_\infty$ leveraging, and this opportunity should not be disregarded.

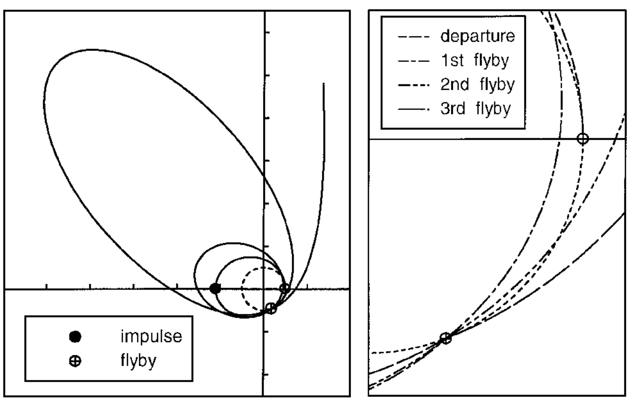


**Fig. 3 $2{:}1^-$ $\Delta V$-EEEGA suboptimal escape trajectory (inertial frame).**

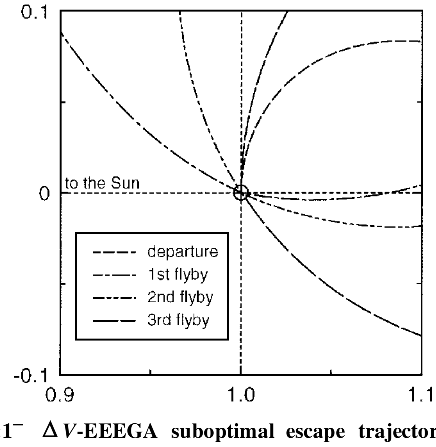


**Fig. 4 $2{:}1^-$ $\Delta V$-EEEGA suboptimal escape trajectory (rotating frame).**

### Multiple $\Delta V$-EGA (M$\Delta V$-EGA)

According to the suggestions of Pontryagin's Maximum Principle, a series of $\Delta V$-EGA maneuvers seems to be the cheapest way of escaping from the two-body (sun-Earth) system. The last $\Delta V$-EGA maneuver provides escape, but the aphelion, where the deep-space impulse is applied, is attained by means of another $\Delta V$-EGA maneuver, that, in turn, could be improved by yet another one. There is no apparent limit to the number of maneuvers; practical limits arise from an acceptable number of flybys (and engine burns) and Earth revolutions between departure and final flyby. In the present paper the number of Earth revolutions between two consecutive flybys is also bounded to five (a lower limit would preclude the possibility of escaping from the solar system); 36 $\Delta V$-EGA maneuvers are therefore considered and are divided into nine groups according to the ratio $K{:}L$. (The 4:2 group is neglected as it does not offer any appreciable advantage with respect to the 2:1 group.) A practical approach preliminarily defines how many and which groups are used to obtain the escape. For a better performance and simpler analysis any aphelion where a deep-space impulse is applied should be attained by means of a free-height (i.e., free-deviation) flyby; this rule is fulfilled throughout this paper. The maneuvers of the same group present similar characteristics; one maneuver out of each group must be selected. No useful criterion has been found to help guide this selection, and an analysis of all of the possible combinations has been carried out. The M$\Delta V$-EGA maneuvers are classified by means of the series of the composing simple $\Delta V$-EGA maneuvers.

The cheapest escape ($\Delta V_{tot} = 4.678$ km/s) has been obtained by using one $\Delta V$-EGA out of each group; the mission requires 36 years and 10 engine burns. Other interesting missions use five $\Delta V$-EGA maneuvers (18 years, 6 burns, $\Delta V_{tot} = 4.786$ km/s) or three $\Delta V$-EGA maneuvers (10 years, 4 burns, $\Delta V_{tot} = 5.237$ km/s). The latter mission is presented in Fig. 5 (inertial reference frame) and Fig. 6 (rotating reference frame). The spacecraft leaves the Earth parallel to the planet's velocity after the first two flybys; the last minimum-height flyby cannot provide the optimal $v_\infty$ direction.

Each 2:1, 3:1, or 5:1 group contains two $\Delta V$-EGA maneuvers, and eight different combinations are possible. The optimization procedure finds the same number of local minima, i.e., trajectories that satisfy the necessary conditions for optimality. The best of these trajectories consists of the same sequence of $2{:}1^-$, $3{:}1^-$, and $5{:}1^+$ $\Delta V$-EGA maneuvers even though the eccentricity of the Earth's orbit is taken into account; the characteristic velocity that is necessary for escaping is reduced to $\Delta V_{tot} = 5.146$ km/s. The corresponding spacecraft trajectory is presented in Fig. 7; departure and final flyby occur when the Earth is very close to the perihelion.

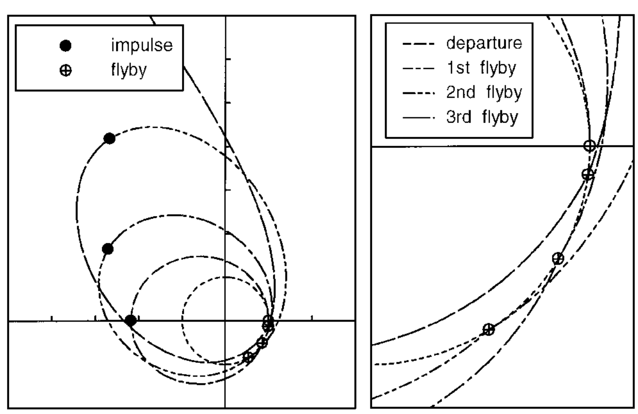


**Fig. 5 $2{:}1^-$ M$\Delta V$-EGA suboptimal escape trajectory (inertial frame).**

The same mission concept can be used to obtain a finite distance from the sun, but an incomplete sequence of maneuvers is sufficient for $r_{max} < 4.85$. Figure 8 is an enlargement of Fig. 2 and provides the $\Delta V$ requirement and the last necessary $\Delta V$-EGA maneuver for an assigned aphelion. One should note that the sequence shown is optimal only for the escape; if a finite distance is sought, marginal improvements could be obtained by selecting different maneuvers from each group, that is, a different combination.

The fuel consumption is further reduced if $\Delta V$-EGA maneuvers with $K > 5$ are also considered. Figures 2 and 8 indicate that the addition of a $K{:}L$ group with $L = K - 1$ moves the initial point of the curve to the left (however, no advantage is obtained for $K > 6$);

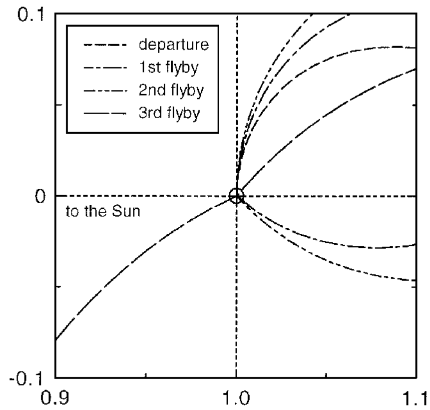


**Fig. 6 2:1$^-$ M$\Delta V$-EGA suboptimal escape trajectory (rotating frame).**

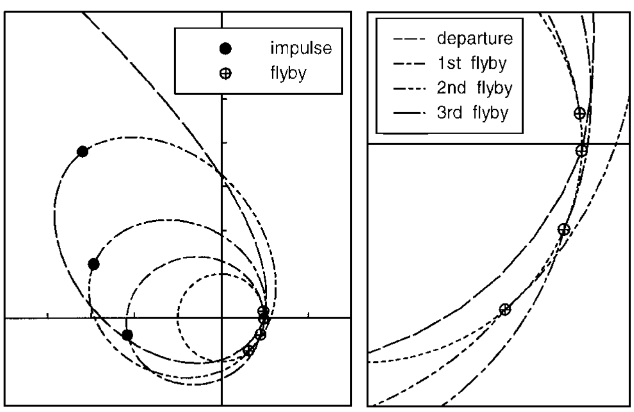


**Fig. 7 2:1$^-$ M$\Delta V$-EGA optimal escape trajectory (elliptic Earth's orbit).**

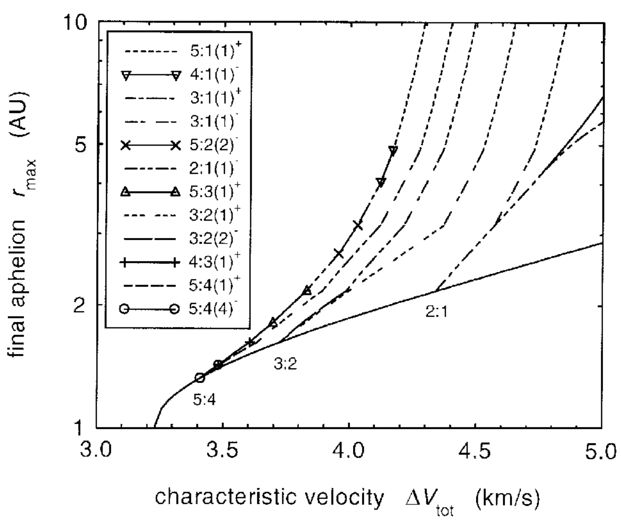


**Fig. 8 Performance of different suboptimal maneuvers based on $\Delta V$-EGA concept.**

a $K{:}L$ maneuver with $L = 1$ produces a steeper final segment of each curve; and the other $K{:}L$ groups ($1 < L < K - 1$) make the curvature of the central portion higher. Not only are mission time and complexity increased, but the number of possible combinations also becomes extremely large.

## Conclusions

Several strategies for escaping from the sun-Earth system have been considered. A simple algebraic procedure provides near-optimal solutions and has permitted an extensive analysis of the problem. The most interesting solutions can be improved by means of an indirect optimization procedure that also suggests how the varying velocity of the Earth on its elliptic orbit can be profitably used.

The fuel consumption required for the escape is progressively reduced if the number of flybys and engine burns, as well as the mission time, is increased. The analysis cannot find a minimum-fuel trajectory unless a practical application suggests useful constraints to the solution.

Practical missions make use of gravity assist from Venus to reach Jupiter and from the outer planets to reach the escape energy. Future work could be based on the two-body problem equations with more planets to analyze other strategies or the three-body problem equations to improve the gravitation model.

## Appendix: Optimization

The optimal conditions for a minimum-fuel $\Delta V$-EGA trajectory have been presented in a previous paper.[7] The same boundary conditions apply for every flyby and deep-space impulse that is performed during a multiple $\Delta V$-EGA trajectory. Some modification is required when an escape from the solar system is sought; in this case it is impractical to keep the escape parabola inside the numerical procedure, but it is convenient to choose the point just before the last flyby (subscript $-$) as the final point of the trajectory and to consider the velocity components after the flyby (subscript $+$) as parameters.

The spacecraft specific energy must be zero after the flyby when the spacecraft enters the escape parabola. This condition completes the constraints of a free-height flyby[6]:

$$v_+^2 - 2/|\boldsymbol{r}_E| = 0 \qquad \boldsymbol{r}_- = \boldsymbol{r}_E(t_-) \qquad \boldsymbol{v}_{\infty-}^2 = \boldsymbol{v}_{\infty+}^2 \tag{A1}$$

where the hyperbolic excess velocities $\boldsymbol{v}_{\infty\pm} = \boldsymbol{v}_\pm - \boldsymbol{v}_E$ have been introduced.

By imposing the necessary conditions for optimality, the adjoint vector $\boldsymbol{\lambda}_{r_-} = -\boldsymbol{\mu}_2$ becomes free; one also obtains

$$\boldsymbol{\lambda}_{v_-} = 2\mu_3 \boldsymbol{v}_{\infty-} \qquad \mu_1 \boldsymbol{v}_+ = \mu_3 \boldsymbol{v}_{\infty+} \tag{A2}$$

i.e., the hyperbolic excess velocity is parallel to the primer vector $\boldsymbol{\lambda}_v$ before the flyby, whereas it is parallel to the sun-relative velocity $\boldsymbol{v}$ (and also to the Earth's velocity $\boldsymbol{v}_E$) after the flyby. The value of the constant $\mu_1 = \lambda_{v_-}/(2v_+)$ is easily found by means of Eqs. (A1) and (A2). The transversality condition concerning the Hamiltonian can be written as

$$\boldsymbol{\lambda}_{r_-} \cdot \boldsymbol{v}_{\infty-} + 2\mu_1 \boldsymbol{v}_{\infty+} \cdot \boldsymbol{g} = 0 \tag{A3}$$

The constraint on the velocity turn $\boldsymbol{v}_{\infty+} \cdot \boldsymbol{v}_{\infty-} = -\cos 2\phi \boldsymbol{v}_{\infty-}^2$ is added in the case of a minimum-height flyby.[7] The hyperbola half-turn $\phi$ is a function only of $\boldsymbol{v}_{\infty-}$ and of the circular velocity at the hyperbola perigee $v_p$

$$\cos\phi = \frac{v_p^2}{\boldsymbol{v}_{\infty-}^2 + v_p^2} \tag{A4}$$

The parameter

$$A = \boldsymbol{v}_{\infty-} \cdot \frac{\mathrm{d}\phi}{\mathrm{d}\boldsymbol{v}_{\infty-}} = \frac{2}{\tan\phi} \frac{v_{\infty-}^2}{v_{\infty-}^2 + v_p^2} \tag{A5}$$

is usefully defined, and the necessary conditions for optimality provide, besides $\boldsymbol{\lambda}_{r_-} = -\boldsymbol{\mu}_2$,

$$2\mu_1 v_+^\perp = \lambda_{v_-}^\perp \qquad 2\mu_1 v_+^\parallel = \lambda_{v_-}^\parallel + 2\lambda_{v_-}^\perp A \tag{A6}$$

where subscripts $\perp$ and $\parallel$ denote the components perpendicular and parallel to the hyperbolic excess velocity, respectively. The transversality condition is again expressed by Eq. (A3), but $\mu_1$ is now provided by Eqs. (A1) and (A6).

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